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Spiralling directed self-avoiding walks

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Abstract. A self-avoiding walk model is proposed, in which the walker performs two-choice directed self-avoiding walks while it may also make spiralling turns, and thus visiting every quarter of the plane unlimited times. The generating function for the square lattice is obtained, and an asymptotic form for the number of N-step walks, a_N, is derived. The a_N of the model belongs to the same universality class with the two-choice directed self-avoiding walk. However, the model achieves a higher number of N-step walks than the two-choice directed self-avoiding walk by a factor of $C \doteq 4.23609\ldots$.

A number of modified self-avoiding walk models with additional constraints have actively been studied. It is hoped that the studies on such modified models may shed light on the properties of the full self-avoiding walk model (SAW), and provides useful clues on how to approach the extremely difficult SAW problem. For example, the study of such modified models may provide some clues on how the changes in the constraints may affect the universality of the model. This is also the motivation of this study. By combining the directed and spiralling self-avoiding walks, it is hoped to learn how the two types of constraints may affect the universality.

Among the modified self-avoiding walk models with additional constraints, the spiralling self-avoiding walk (SSAW) ([1, 2] and references therein) and directed self-avoiding walk (DSAW) ([3, 4] and references therein) are studied most and understood quite well. The two models belong to different universality classes. As a matter of fact, the SSAW exhibits different universality in square and triangular latties ([5] and references therein). In SSAW, the walker may visit all the quarters of the plane *spiralling around* without any limitation, whereas the walker in DSAW is restricted to certain quarters—one quarter in two-choice DSAW and two quarters in three-choice DSAW. DSAW exhibits different scaling behaviour in the parallel and perpendicular directions to the preferred axis of the walks ([3] and references therein). Such an anisotropy is removed in a model proposed by the author and co-workers [6, 7]. The model is a generalization of the two- and three-choice DSAWs and the walker may reach all the quarters of the plane. However, *spiralling around* is not allowed in the model, and naturally the universality of the model is the same with the DSAWs (but without anisotropy).

In the present model, the walker performs two-choice DSAW with unrestricted *spiralling around*. Let us take a walker that starts out from the origin in the (x_+) direction (in the following, we always assume that the walks start out in the (x_+) direction). Aftr the initial (x_+) step, the walker may take either the (x_+) or the (y_+) direction, performing the

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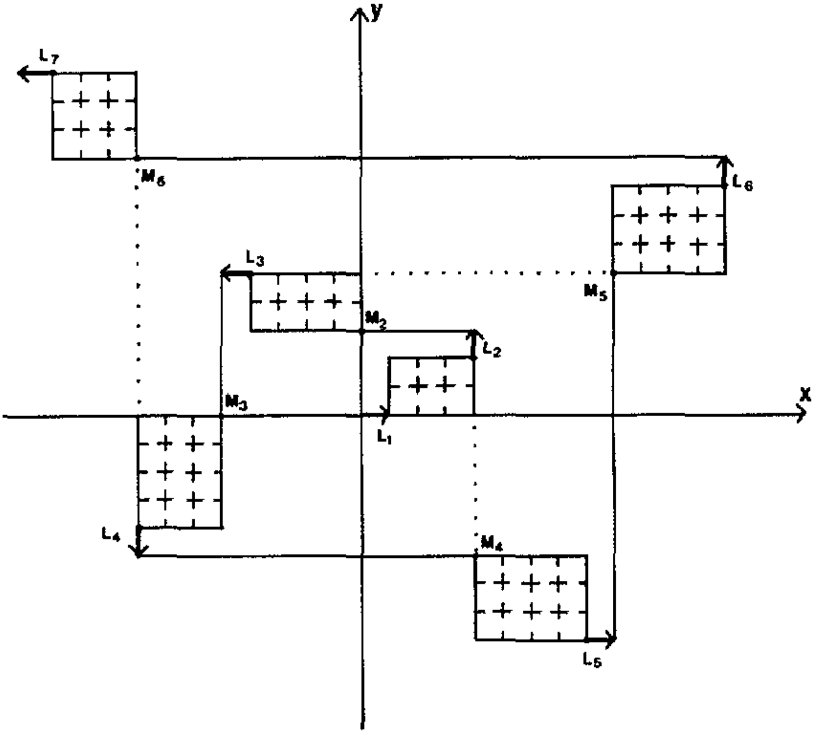

Figure 1. The walker starts out in the (x_+) direction and thereafter may take the (x_+) or (y_+) directions, performing a two-choice DSAW in the (x_+, y_+) directions (shown as a rectangle) to a point L_2. After a (y_+) step, the walker may make a *spiralling turn* taking the (x_-) direction. Notice that, to make a self-avoiding walk in the (x_-) direction, the walker should have ended in a (y_+) step (bold arrow at L_2), just before the *spiralling turn*. The walker keeps taking only (x_-) steps until it sees the end of the previous runs at its left (until it reaches M_2), and then may begin to perform a two-choice DSAW in the (y_+, x_-) directions to a point L_3. Just after the (x_-) step (bold arrow at L_3), the walker may make another *spiralling turn* taking a (y_-) step and keep going in that direction until it sees the end of the previous runs at its left (until it reaches M_3). From the point M_3 on, it may begin to perform a two-choice DSAW in the (x_-, y_-) directions. Thereafter, in similar fashions, the walker may make a *spiralling turn* and perform two-choice DSAWs in the sequence of (y_-, x_+), (x_+, y_+) directions and so forth, but it can begin to perform the DSAW only after it reaches the ends of the previous runs to its left after each *spiralling turn*.

two-choice DSAW in (x_+, y_+) directions. Next, the walker may make a turn by taking the (x_-) direction (we shall call this type of turn a *spiralling turn* to distinguish it from the turns in DSAW). We note here that, to have a self-avoiding walk, the walker should have ended in a (y_+) step just prior to taking the *spiralling turn* to the (x_-) direction. After this *spiralling turn*, we impose an additional restriction. The walker cannot make any turn but keeps going straight on in the (x_-) direction until it sees the end of the previous runs on its left. On reaching the end of the previous runs, the walker may perform two-choice DSAW in the (y_+, x_-) directions. Thereafter, in a similar fashion, the walker may perform two-choice DSAW in the sequence of (x_-, y_-), (y_-, x_+), (x_+, y_+) and (y_+, x_-) directions and so forth, *spiralling around* unlimited times. But, after each *spiralling turn*, the walker may perform the two-choice DSAWs only after seeing the end of the previous runs on its left (see figure 1). This additional restriction on the initiation of the two-choice DSAW is neither natural nor desired. However, without this restriction, the resulting generating function shall contain binomial coefficients with *mixed variables*, and it is extremely difficult to manage it analytically. The generating function of the walk is given by

$$G = t_1 + t_1(t_2 + r_2) + t_1 t_2(t_3 + r_3) + t_1 t_2 t_3(t_4 + r_4) + \cdots \tag{1a}$$

where

$$t_1 = x_+ \sum_{m_1, n_1=0}^{\infty} \binom{m_1 + n_1}{m_1} x_+^{m_1} y_+^{n_1} \tag{1b}$$

$$t_2 = y_+ x_-^{m_1+1} \sum_{m_2, n_2=0}^{\infty} \binom{m_2 + n_2}{m_2} x_-^{m_2} y_+^{n_2} \tag{1c}$$

$$t_3 = x_- y_-^{n_1+n_2+1} \sum_{m_3, n_3=0}^{\infty} \binom{m_3 + n_3}{m_3} x_-^{m_3} y_-^{n_3} \tag{1d}$$

$$t_4 = y_- x_+^{m_1+m_2+m_3+2} \sum_{m_4, n_4=0}^{\infty} \binom{m_4 + n_4}{m_4} x_+^{m_4} y_-^{n_4} \tag{1e}$$

$$t_5 = x_+ y_+^{n_1+\cdots+n_4+2} \sum_{m_5, n_5=0}^{\infty} \binom{m_5 + n_5}{m_5} x_+^{m_5} y_+^{n_5} \tag{1f}$$

$$\vdots$$

and

$$r_2 = y_+(x_- + x_-^2 + \cdots + x_-^{m_1+1}) = y_+ \frac{x_- - x_-^{m_1+2}}{1 - x_-} \tag{1g}$$

$$r_3 = x_-(y_- + y_-^2 + \cdots + y_-^{n_1+n_2+1}) = x_- \frac{y_- - y_-^{n_1+n_2+2}}{1 - y_-} \tag{1h}$$

$$r_4 = y_-(x_+ + x_+^2 + \cdots + x_+^{m_1+m_2+m_3+2}) = y_- \frac{x_+ - x_+^{m_1+m_2+m_3+3}}{1 - x_+} \tag{1i}$$

$$r_5 = x_+(y_+ + y_+^2 + \cdots + y_+^{n_1+\cdots+n_4+2}) = x_+ \frac{y_+ - y_+^{n_1+\cdots+n_4+3}}{1 - y_+} \tag{1j}$$

$$\vdots$$

Here, the r_i's represent the walks in which the last stretch of the walks does not go beyond the previous runs. Putting $z = x_\pm = y_\pm$, we have

$$G = \sum_{L=1}^{\infty} T_L + \sum_{L=1}^{\infty} R_L \tag{2a}$$

where

$$T_L \equiv \prod_{l=1}^{L} f_l(z) \tag{2b}$$

$$f_{2l-1}(z) \equiv \frac{z^l}{1-(z^l+z^l)} \tag{2c}$$

$$f_{2l}(z) \equiv \frac{z^l}{1-(z^l+z^{l+1})} \tag{2d}$$

and

$$R_L \equiv \begin{cases} 0 & \text{for } L=1 \\ \dfrac{z}{1-z}\left(T_{L-1}-\dfrac{T_L}{T_1}\right) & \text{for } L \geqslant 2. \end{cases} \tag{2e}$$

Using Rouché's theorem, it can be shown that the singularity of $f_l(z)$ closest to the origin is the simple pole at $z=\frac{1}{2}$. We notice that the singularity of SSAW at $z=1$ [1] is not apparent here. The walks of SSAW type are dominated by the DSAW type and they are obscured in the expression. The simple pole at $z=\frac{1}{2}$ is the singularity of the DSAW-type walks [3], and it is the most important singularity here.

To obtain the number of N-step walks, a_N, we evaluate

$$c_N(L) \equiv \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}z}{z^{N+1}} T_L(z). \tag{3}$$

For an asymptotic expression, only the pole at $z=\frac{1}{2}$ is relevant. The results are,

$$c_N(1) = 2^{N-1} \tag{4a}$$

$$C_N(2L-1) = 2^{N-1} \prod_{k=2}^{2L-1} \frac{1}{(2^k-2)(2^k-3)} \qquad \text{for } L \geqslant 2 \tag{4b}$$

$$c_N(2) = 2^{N-1} \tag{4c}$$

and

$$c_N(2L) = 2^{N-1} \prod_{k=2}^{2L} \frac{1}{(2^k-2)(2^{k+1}-3)} \qquad \text{for } L \geqslant 2. \tag{4d}$$

In the evaluation of

$$d_N(L) \equiv \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}z}{z^{N+1}} \frac{z}{1-z} R_L(z) \tag{5}$$

we notice that the T_L/T_1 term in R_L has no singularity at $z=\frac{1}{2}$, and the term may be neglected. We obtain

$$d_N(2L-1) = 2^{N-1} \prod_{k=2}^{2L-1} \frac{1}{(2^k-2)(2^k-3)} \qquad \text{for } L \geqslant 2 \tag{6a}$$

$$d_N(2) = 2^{N-1} \tag{6b}$$

and

$$d_N(2L) = 2^{N-1} \prod_{k=2}^{2L} \frac{1}{(2^k - 2)(2^{k+1} - 3)} \qquad \text{for } L \geqslant 2. \tag{6c}$$

Summing all these results, we get

$$a_N = C2^{N-1}$$

where

$$C = 3 + 2\sum_{L=2}^{\infty}\left[\prod_{k=2}^{L}\frac{1}{(2^k-2)(2^k-3)} + \prod_{k=2}^{L}\frac{1}{(2^k-2)(2^{k+1}-3)}\right]$$
$$\fallingdotseq 4.236\,09\ldots. \tag{7}$$

Actually, the upper limit of the summation in equation (7) is $O(\sqrt{N})$. However, the series decreases so rapidly that the limit may be replaced by infinity.

The walk is fundamentally a DSAW type. The *spiralling around* does not change the universality since the number of N-step walks decreases too rapidly as the number of the *spiralling turns*, L, increases. This is due to the restriction on their initiation of DSAW after each *spiralling turn*. The restriction makes the number of N-step walks a monotonically fast decreasing function of L. If we remove this restriction and allow DSAW all the time, the number of N-step walks may have a peak at a certain value of $L > 1$ and also the L-dependency may change, and the model may show a different universality.

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